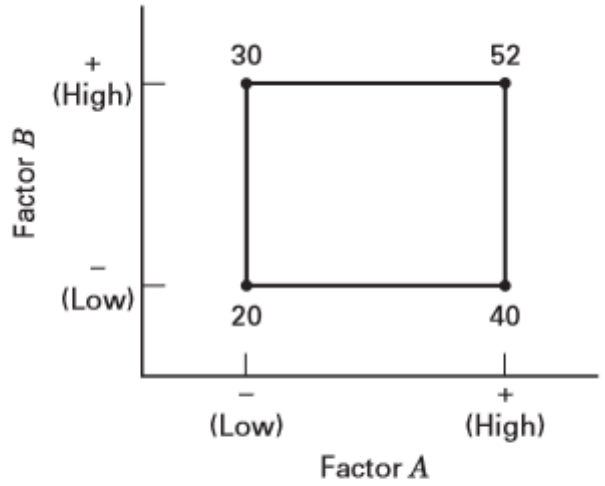
Introduction to Factorial Designs

- Many experiments involve the study of the effects of two or more factors. In general, factorial designs are most efficient for this type of experiment.
- In each complete trial or replicate of the experiment all possible combinations of the levels of the factors are investigated.
- For example, if there are a levels of factor A and b levels of factor B, each replicate contains all ab treatment combinations. When factors are arranged in a factorial design, they are often said to be crossed.
- The effect of a factor is defined to be the change in response produced by a change in the level of the factor. This is frequently called a main effect because it refers to the primary factors of interest in the experiment.

A 2 actor experiment



■ **FIGURE 5.1** **A two-factor factorial experiment, with the response (y) shown at the corners**

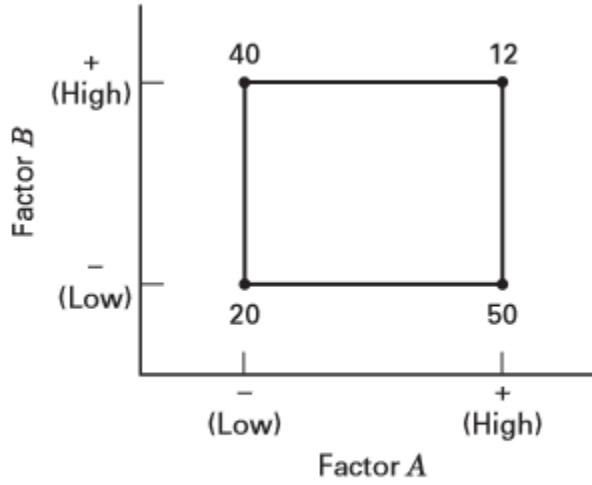
- This is a two-factor factorial experiment with both design factors at two levels. We have called these levels "low" and "high" and denoted them "+" and "-" respectively. The main effect of factor A in this two-level design can be thought of as the difference between the average response at the low level of A and the average response at the high level of A. Numerically, this is

$$A = \frac{40 + 52}{2} - \frac{20 + 30}{2} = 21$$

- That is, increasing factor A from the low level to the high level causes an average response increase of 21 units. Similarly, the main effect of B is

$$B = \frac{30 + 52}{2} - \frac{20 + 40}{2} = 11$$

Interaction



■ **FIGURE 5.2** **A two-factor factorial experiment with interaction**

- In some experiments, we may find that the difference

in response between the levels of one factor is not the same at all levels of the other factors. When this occurs, there is an interaction between the factors.

At the low level of factor B (or B^-), the A effect is

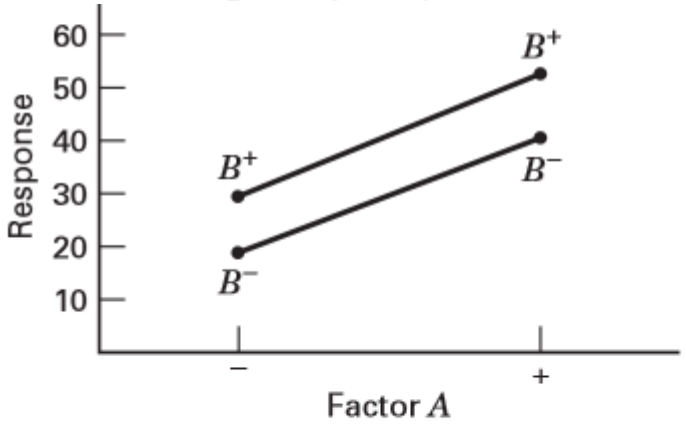
$$A = 50 - 20 = 30$$

and at the high level of factor B (or B^+), the A effect is

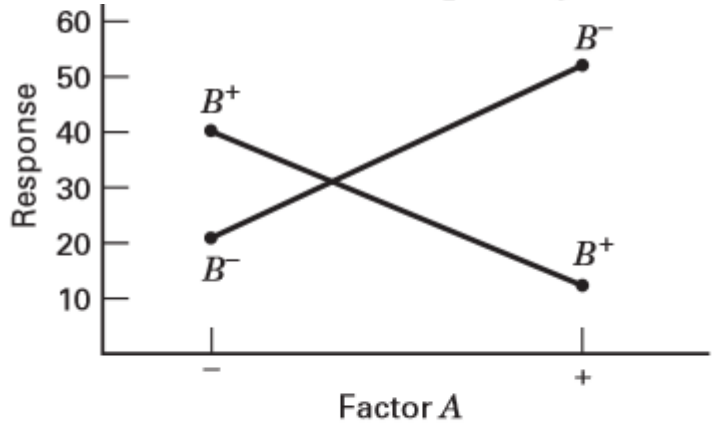
$$A = 12 - 40 = -28$$

Because the effect of A depends on the level chosen for factor B, we see that there is interaction between A and B. The magnitude of the interaction effect is the average *difference* in these two A effects, or $AB = (-28 - 30)/2 = -29$. Clearly, the interaction is large in this experiment.

Figure 5.3 plots the response data in Figure 5.1 against factor A for both levels of factor B. Note that the B^- and B^+ lines are approximately parallel, indicating a lack of interaction between factors A and B. Similarly, Figure 5.4 plots the response data in Figure 5.2. Here we see that the B^- and B^+ lines are not parallel. This indicates an interaction between factors A and B. Two-factor interaction graphs such as these are frequently very useful in interpreting significant interactions and in reporting results



■ **FIGURE 5.3** **A factorial experiment without interaction**



■ **FIGURE 5.4** **A factorial experiment with interaction**

- There is another way to illustrate the concept of interaction. Suppose that both of our design factors are quantitative (such as temperature, pressure, time, etc.). Then a regression model representation of the two-factor factorial experiment could be written as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

where y is the response, the β's are parameters whose values are to be determined, x_1 is a variable that represents factor A, x_2 is a variable that represents factor B, and ϵ is a random error term. The variables x_1 and x_2 are defined on a **coded scale** from -1 to $+1$ (the low and high levels of A and B), and $x_1 x_2$ represents the interaction between x_1 and x_2.

The parameter estimates in this regression model turn out to be related to the effect estimates. For the experiment shown in Figure 5.1 we found the main effects of A and B to be $A = 21$ and $B = 11$. The estimates of β_1 and β_2 are one-half the value of the corresponding main effect; therefore, $\hat{\beta}_1 = 21/2 = 10.5$ and $\hat{\beta}_2 = 11/2 = 5.5$. The interaction effect in Figure 5.1 is $AB = 1$, so the value of interaction coefficient in the regression model is $\hat{\beta}_{12} = 1/2 = 0.5$. The parameter β_0 is estimated by the average of all four responses, or $\hat{\beta}_0 = (20 + 40 + 30 + 52)/4 = 35.5$. Therefore, the fitted regression model is

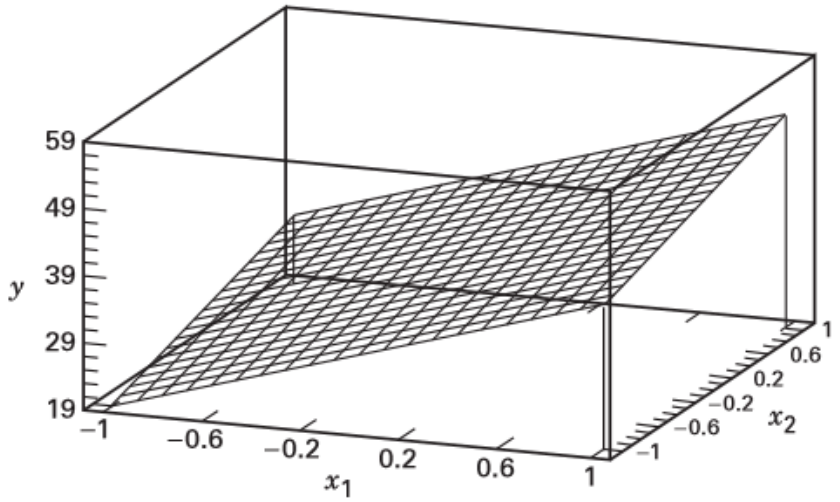
$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 0.5x_1x_2$$

The parameter estimates obtained in the manner for the factorial design with all factors at two levels ($-$ and $+$) turn out to be **least squares estimates** (more on this later).

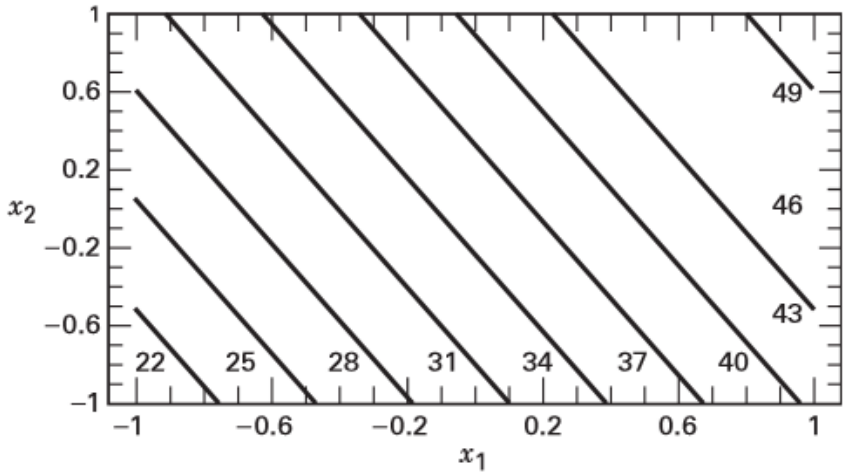
The interaction coefficient ($\hat{\beta}_{12} = 0.5$) is small relative to the main effect coefficients $\hat{\beta}_1$ and $\hat{\beta}_2$. We will take this to mean that interaction is small and can be ignored. Therefore, dropping the term $0.5x_1x_2$ gives us the model

$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2$$

Figure 5.5 presents graphical representations of this model. In Figure 5.5*a* we have a plot of the plane of y-values generated by the various combinations of x_1 and x_2. This three-dimensional graph is called a **response surface plot**. Figure 5.5*b* shows the contour lines of constant response y in the x_1, x_2 plane. Notice that because the response surface is a plane, the contour plot contains parallel straight lines.



(*a*) The response surface



(*b*) The contour plot

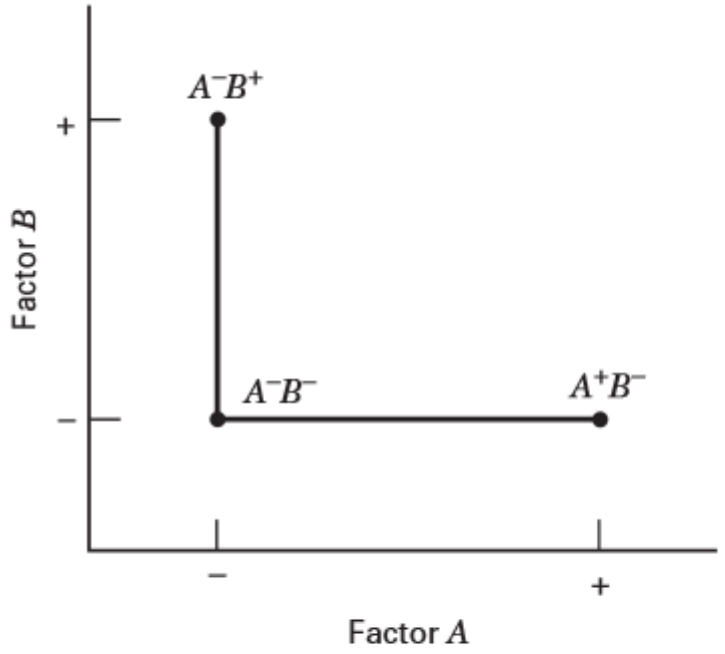
■ **FIGURE 5.5** **Response surface and contour plot for the model $\hat{y} = 35.5 + 10.5x_1 + 5.5x_2$**

Advantages of factors

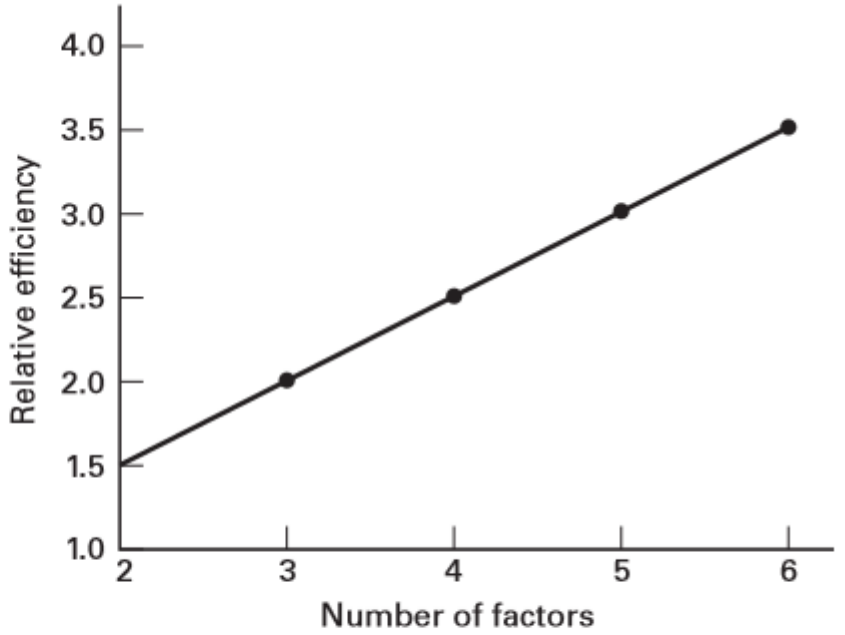
The advantage of factorial designs can be easily illustrated. Suppose we have two factors A and B, each at two levels. We denote the levels of the factors by A^-, A^+, B^-, and B^+. Information on both factors could be obtained by varying the factors one at a time, as shown in Figure 5.7. The effect of changing factor A is given by $A^+B^- - A^-B^-$, and the effect of changing factor B is given by $A^-B^+ - A^-B^-$. Because experimental error is present, it is desirable to take two observations, say, at each treatment combination and estimate the effects of the factors using average responses. Thus, a total of six observations are required.

If a factorial experiment had been performed, an additional treatment combination, A^+B^+, would have been taken. Now, using just *four* observations, two estimates of the A effect can be made: $A^+B^- - A^-B^-$ and $A^+B^+ - A^-B^+$. Similarly, two estimates of the B effect can be made. These two estimates of each main effect could be averaged to produce average main effects that are *just as precise* as those from the single-factor experiment, but only four total observations are required and we would say that the relative efficiency of the factorial design to the one-factor-at-a-time experiment is (6/4) = 1.5. Generally, this relative efficiency will increase as the number of factors increases, as shown in Figure 5.8.

Now suppose interaction is present. If the one-factor-at-a-time design indicated that A^-B^+ and A^+B^- gave better responses than A^-B^-, a logical conclusion would be that A^+B^+ would be even better. However, if interaction is present, this conclusion may be *seriously in error*.



■ **FIGURE 5.7** A one-factor-at-a-time experiment



■ **FIGURE 5.8** Relative efficiency of a factorial design to a one-factor-at-a-time experiment (two factor levels)

Two factor factorial design

- The simplest types of factorial designs involve only two factors or sets of treatments.
- There are a levels of factor A and b levels of factor B, and these are arranged in a factorial design; that is, each replicate of the experiment contains all ab treatment combinations. In general, there are n replicates.
- Consider an engineer is designing a battery for use in a device that will be subjected to some extreme variations in temperature. There are three choices of plate material from which he has to select one.
- The temperature can be controlled in the product development laboratory for the purposes of a test.
- The engineer decides to test all three plate materials at three temperature levels—15, 70, and 125°F—because these temperature levels are consistent with the product end-use environment. Because there are two factors at three levels, this design is sometimes called a 3^2 factorial design.

- Four batteries are tested at each combination of plate material and temperature, and all 36 tests are run in random order.

■ TABLE 5.1
Life (in hours) Data for the Battery Design Example

Material Type	Temperature (°F)					
	15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

In this problem the engineer wants to answer the following questions:

1. What effects do material type and temperature have on the life of the battery?
2. Is there a choice of material that would give *uniformly long life regardless of temperature*?

This last question is particularly important. It may be possible to find a material alternative that is not greatly affected by temperature. If this is so, the engineer can make the battery **robust** to temperature variation in the field. This is an example of using statistical experimental design for **robust product design**, a very important engineering problem.

This design is a specific example of the general case of a two-factor factorial. To pass to the general case, let y_{ijk} be the observed response when factor A is at the ith level ($i = 1, 2, \ldots, a$) and factor B is at the jth level ($j = 1, 2, \ldots, b$) for the kth replicate ($k = 1, 2, \ldots, n$). In general, a two-factor factorial experiment will appear as in Table 5.2. The order in which the abn observations are taken is selected at random so that this design is a **completely randomized design**.

- The observations in a factorial experiment can be described by a model.
- The effects model is:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \qquad \textbf{(5.1)}$$

where μ is the overall mean effect, τ_i is the effect of the ith level of the row factor A, β_j is the effect of the jth level of column factor B, $(\tau\beta)_{ij}$ is the effect of the interaction between τ_i and β_j, and ϵ_{ijk} is a random error component. Both factors are assumed to be **fixed**, and the treatment effects are defined as deviations from the overall mean, so $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$. Similarly, the interaction effects are fixed and are defined such that $\sum_{i=1}^{a} (\tau\beta)_{ij} = \sum_{j=1}^{b} (\tau\beta)_{ij} = 0$. Because there are n replicates of the experiment, there are abn total observations.

- The means model:

Another possible model for a factorial experiment is the **means model**

$$y_{ijk} = \mu_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

where the mean of the ijth cell is

$$\mu_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij}$$

- In the two-factor factorial, both row and column factors (or treatments), A and B, are of equal interest.
- Testing equality of row effects:

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1: \text{at least one } \tau_i \neq 0$$

- Testing equality of column effects:

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_b = 0$$
$$H_1: \text{at least one } \beta_j \neq 0$$

- Testing interaction effects:

$$H_0: (\tau\beta)_{ij} = 0 \qquad \text{for all } i, j$$
$$H_1: \text{at least one } (\tau\beta)_{ij} \neq 0$$

Statistical Analysis of the Fixed Effects Mode

Let $y_{i..}$ denote the total of all observations under the ith level of factor A, $y_{.j.}$ denote the total of all observations under the jth level of factor B, $y_{ij.}$ denote the total of all observations in the ijth cell, and $y_{...}$ denote the grand total of all the observations. Define $\bar{y}_{i..}$, $\bar{y}_{.j.}$, $\bar{y}_{ij.}$, and $\bar{y}_{...}$ as the corresponding row, column, cell, and grand averages. Expressed mathematically,

$$y_{i..} = \sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{i..} = \frac{y_{i..}}{bn} \qquad i = 1, 2, \ldots, a$$

$$y_{.j.} = \sum_{i=1}^{a}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{.j.} = \frac{y_{.j.}}{an} \qquad j = 1, 2, \ldots, b$$

$$y_{ij.} = \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{ij.} = \frac{y_{ij.}}{n} \qquad \begin{matrix} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{matrix}$$

$$y_{...} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{...} = \frac{y_{...}}{abn} \qquad (5.3)$$

The **total corrected sum of squares** may be written as

$$\begin{aligned}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2 &= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}[(\bar{y}_{i..}-\bar{y}_{...})+(\bar{y}_{.j.}-\bar{y}_{...})\\ &\quad+(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})+(y_{ijk}-\bar{y}_{ij.})]^2\\ &= bn\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2+an\sum_{j=1}^{b}(\bar{y}_{.j.}-\bar{y}_{...})^2\\ &\quad+n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2\\ &\quad+\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2 \qquad (5.4)\end{aligned}$$

because the six cross products on the right-hand side are zero. Notice that the total sum of squares has been partitioned into a sum of squares due to "rows," or factor A, (SS_A); a sum of squares due to "columns," or factor B, (SS_B); a sum of squares due to the interaction between A and B, (SS_{AB}); and a sum of squares due to error, (SS_E). This is the fundamental ANOVA equation for the two-factor factorial. From the last component on the right-hand side of Equation 5.4, we see that there must be at least two replicates ($n \geq 2$) to obtain an error sum of squares.

We may write Equation 5.4 symbolically as

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

The number of degrees of freedom associated with each sum of squares is

Effect	Degrees of Freedom
A	$a - 1$
B	$b - 1$
AB interaction	$(a - 1)(b - 1)$
Error	$ab(n - 1)$
Total	$abn - 1$

- The main effects A and B have a and b levels, respectively; therefore they have a - 1 and b - 1 degrees of freedom as shown. The interaction degrees of freedom are simply the number of degrees of freedom for cells (which is ab - 1) minus the number of degrees of freedom for the two main effects A and B; that is, ab - 1 - (a - 1) - (b 1) = (a-1)(b-1)

- Within each of the ab cells, there are n - 1 degrees of freedom between the n replicates; thus there are ab(n - 1) degrees of freedom for error.

Each sum of squares divided by its degrees of freedom is a **mean square**. The expected values of the mean squares are

$$E(MS_A) = E\left(\frac{SS_A}{a-1}\right) = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1}$$

$$E(MS_B) = E\left(\frac{SS_B}{b-1}\right) = \sigma^2 + \frac{an\sum_{j=1}^{b}\beta_j^2}{b-1}$$

$$E(MS_{AB}) = E\left(\frac{SS_{AB}}{(a-1)(b-1)}\right) = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

and

$$E(MS_E) = E\left(\frac{SS_E}{ab(n-1)}\right) = \sigma^2$$

each of the ratios of mean squares MS_A/MS_E, MS_B/MS_E, and MS_{AB}/MS_E is distributed as F with $a - 1$, $b - 1$, and $(a - 1)(b - 1)$ numerator degrees of freedom, respectively, and $ab(n - 1)$ denominator degrees of freedom,[1] and the critical region would be the upper tail of the F distribution. The test procedure is usually summarized in an **analysis of variance table**, as shown in Table 5.3.

■ **TABLE 5.3**

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a - 1$	$MS_A = \frac{SS_A}{a - 1}$	$F_0 = \frac{MS_A}{MS_E}$
B treatments	SS_B	$b - 1$	$MS_B = \frac{SS_B}{b - 1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a - 1)(b - 1)$	$MS_{AB} = \frac{SS_{AB}}{(a - 1)(b - 1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n - 1)$	$MS_E = \frac{SS_E}{ab(n - 1)}$	
Total	SS_T	$abn - 1$		

- One could write out the individual elements of the ANOVA identity

$$y_{ijk} - \bar{y}_{...} = (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.})$$

- The total sum of squares is computed as usual by $SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn}$

The sums of squares for the main effects are

$$SS_A = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn} \qquad (5.7)$$

and

$$SS_B = \frac{1}{an}\sum_{j=1}^{b} y_{.j.}^2 - \frac{y_{...}^2}{abn} \qquad (5.8)$$

It is convenient to obtain the SS_{AB} in two stages. First we compute the sum of squares between the *ab* cell totals, which is called the sum of squares due to "subtotals":

$$SS_{\text{Subtotals}} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{...}^2}{abn}$$

This sum of squares also contains SS_A and SS_B. Therefore, the second step is to compute SS_{AB} as

$$SS_{AB} = SS_{\text{Subtotals}} - SS_A - SS_B \tag{5.9}$$

We may compute SS_E by subtraction as

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B \tag{5.10}$$

or

$$SS_E = SS_T - SS_{\text{Subtotals}}$$

An example

Table 5.4 presents the effective life (in hours) observed in the battery design example described in Section 5.3.1. The row and column totals are shown in the margins of the table, and the circled numbers are the cell totals.

■ **TABLE 5.4**

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i.}$
1	130 74	155 180	(539)	34 80	40 75	(229)	20 82	70 58	(230)	998
2	150 159	188 126	(623)	136 106	122 115	(479)	25 58	70 45	(198)	1300
3	138 168	110 160	(576)	174 150	120 139	(583)	96 82	104 60	(342)	1501
$y_{.j.}$		1738			1291			770		$3799 = y_{...}$

Using Equations 5.6 through 5.10, the sums of squares are computed as follows:

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn}$$

$$= (130)^2 + (155)^2 + (74)^2 + \cdots + (60)^2 - \frac{(3799)^2}{36} = 77{,}646.97$$

$$SS_{\text{Material}} = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn}$$

$$= \frac{1}{(3)(4)}[(998)^2 + (1300)^2 + (1501)^2] - \frac{(3799)^2}{36} = 10{,}683.72$$

$$SS_{\text{Temperature}} = \frac{1}{an}\sum_{j=1}^{b} y_{.j.}^2 - \frac{y_{...}^2}{abn}$$

$$= \frac{1}{(3)(4)}[(1738)^2 + (1291)^2 + (770)^2] - \frac{(3799)^2}{36} = 39{,}118.72$$

$$SS_{\text{Interaction}} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{...}^2}{abn} - SS_{\text{Material}} - SS_{\text{Temperature}}$$

$$= \frac{1}{4}[(539)^2 + (229)^2 + \cdots + (342)^2] - \frac{(3799)^2}{36} - 10{,}683.72 - 39{,}118.72 = 9613.78$$

$$SS_E = SS_T - SS_{\text{Material}} - SS_{\text{Temperature}} - SS_{\text{Interaction}}$$

$$= 77{,}646.97 - 10{,}683.72 - 39{,}118.72 - 9613.78 = 18{,}230.75$$

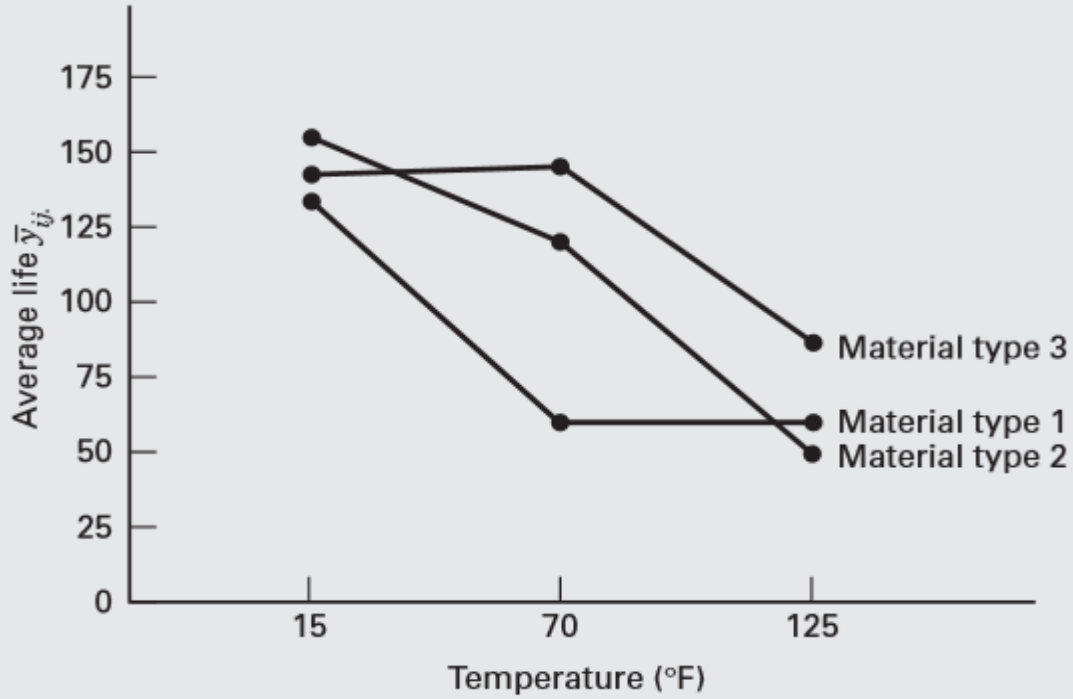
The ANOVA is shown in Table 5.5. Because $F_{0.05,4,27} = 2.73$, we conclude that there is a significant interaction between material types and temperature. Furthermore, $F_{0.05,2,27} = 3.35$, so the main effects of material type and temperature are also significant. Table 5.5 also shows the *P*-values for the test statistics.

■ **TABLE 5.5**

Analysis of Variance for Battery Life Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Material types	10,683.72	2	5,341.86	7.91	0.0020
Temperature	39,118.72	2	19,559.36	28.97	< 0.0001
Interaction	9,613.78	4	2,403.44	3.56	0.0186
Error	18,230.75	27	675.21		
Total	77,646.97	35			

To assist in interpreting the results of this experiment, it is helpful to construct a graph of the average responses at each treatment combination. This graph is shown in Figure 5.9. The significant interaction is indicated by the lack of parallelism of the lines. In general, longer life is attained at low temperature, regardless of material type. Changing from low to intermediate temperature, battery life with material type 3 may actually increase, whereas it decreases for types 1 and 2. From intermediate to high temperature, battery life decreases for material types 2 and 3 and is essentially unchanged for type 1. Material type 3 seems to give the best results if we want less loss of effective life as the temperature changes.



■ **FIGURE 5.9** **Material type–temperature plot for Example 5.1**

Multiple Comparisons

- We perform Tukey's test on battery life data.
- Significant interaction is noted in the experiment
- When interaction is significant, comparisons between the means of one factor (e.g., A) may be obscured by the AB interaction. One approach to this situation is to fix factor B at a specific level and apply Tukey's test to the means of factor A at that level.
- suppose that in Example 5.1 we are interested in detecting differences among the means of the three material types. Because interaction is significant, we make this comparison at just one level of temperature, say level 2 (70°F). We assume that the best estimate of the error variance is the MS_E from the ANOVA table, utilizing the assumption that the experimental error variance is the same over all treatment combinations.

The three material type averages at 70°F arranged in ascending order are

$$\bar{y}_{12.} = 57.25 \qquad \text{(material type 1)}$$
$$\bar{y}_{22.} = 119.75 \qquad \text{(material type 2)}$$
$$\bar{y}_{32.} = 145.75 \qquad \text{(material type 3)}$$

and

$$T_{0.05} = q_{0.05}(3, 27)\sqrt{\frac{MS_E}{n}}$$
$$= 3.50\sqrt{\frac{675.21}{4}}$$
$$= 45.47$$

where we obtained $q_{0.05}(3, 27) \simeq 3.50$ by interpolation in Appendix Table VII. The pairwise comparisons yield

$$\text{3 vs. 1:} \qquad 145.75 - 57.25 = 88.50 > T_{0.05} = 45.47$$
$$\text{3 vs. 2:} \qquad 145.75 - 119.75 = 26.00 < T_{0.05} = 45.47$$
$$\text{2 vs. 1:} \qquad 119.75 - 57.25 = 62.50 > T_{0.05} = 45.47$$

This analysis indicates that at the temperature level 70°F, the mean battery life is the same for material types 2 and 3, and that the mean battery life for material type 1 is significantly lower in comparison to both types 2 and 3.

If interaction is significant, the experimenter could compare *all ab* cell means to determine which ones differ significantly. In this analysis, differences between cell means include interaction effects as well as both main effects. In Example 5.1, this would give 36 comparisons between all possible pairs of the nine cell means.

Model adequacy checking

- The primary diagnostic tool is residual analysis(we use the same battery data, residuals shown in next slide). The residuals for the two-factor factorial model with

$$e_{ijk} = y_{ijk} - \hat{y}_{ijk} \qquad (5.11)$$

and because the fitted value $\hat{y}_{ijk} = \bar{y}_{ij.}$ (the average of the observations in the ijth cell), Equation 5.11 becomes

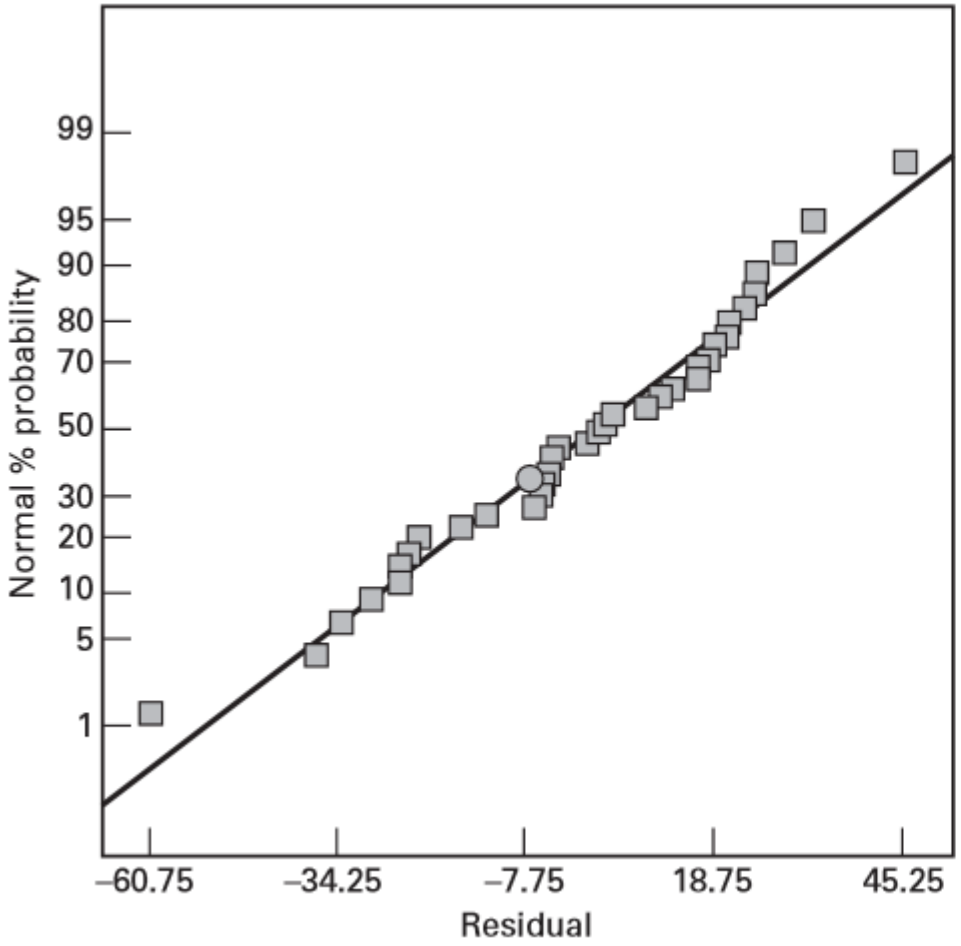
$$e_{ijk} = y_{ijk} - \bar{y}_{ij.} \qquad (5.12)$$

- The normal probability plot of these residuals (Figure 5.11) does not reveal anything particularly troublesome, although the largest negative residual (-60.75 at 15°F for material type 1) does stand out somewhat from the others. The standardized value of this residual is $-60.75/\sqrt{675.21} = -2.34$, and this is the only residual whose absolute value is larger than 2.

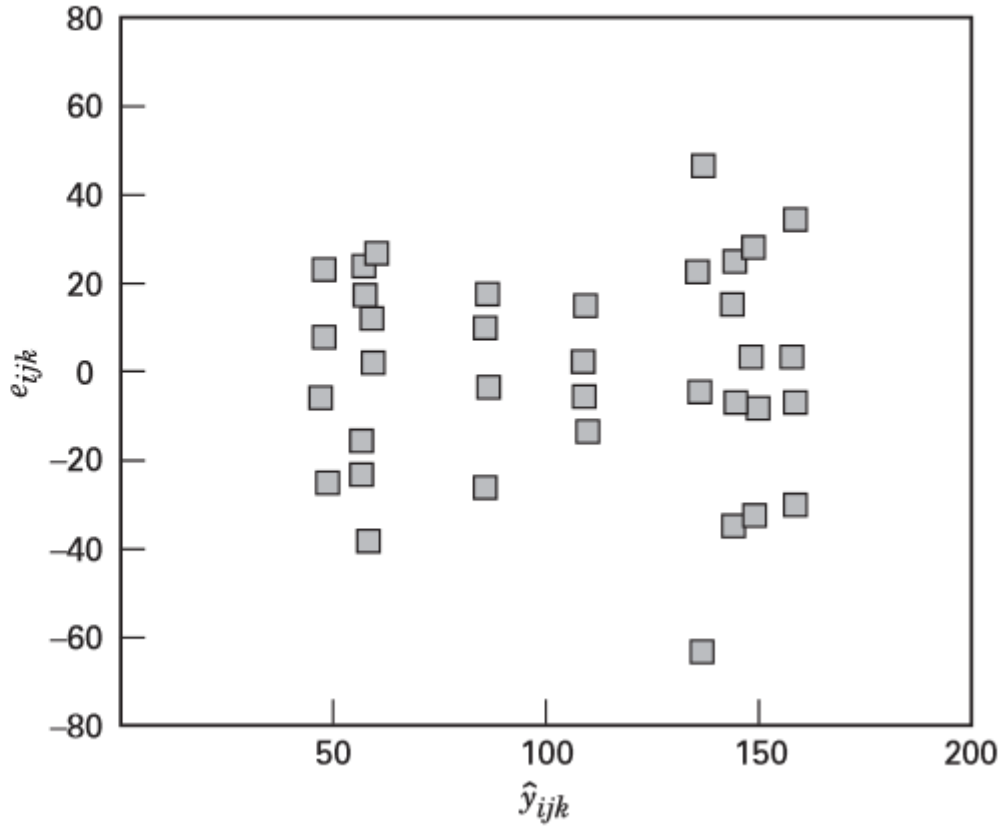
■ TABLE 5.6

Residuals for Example 5.1

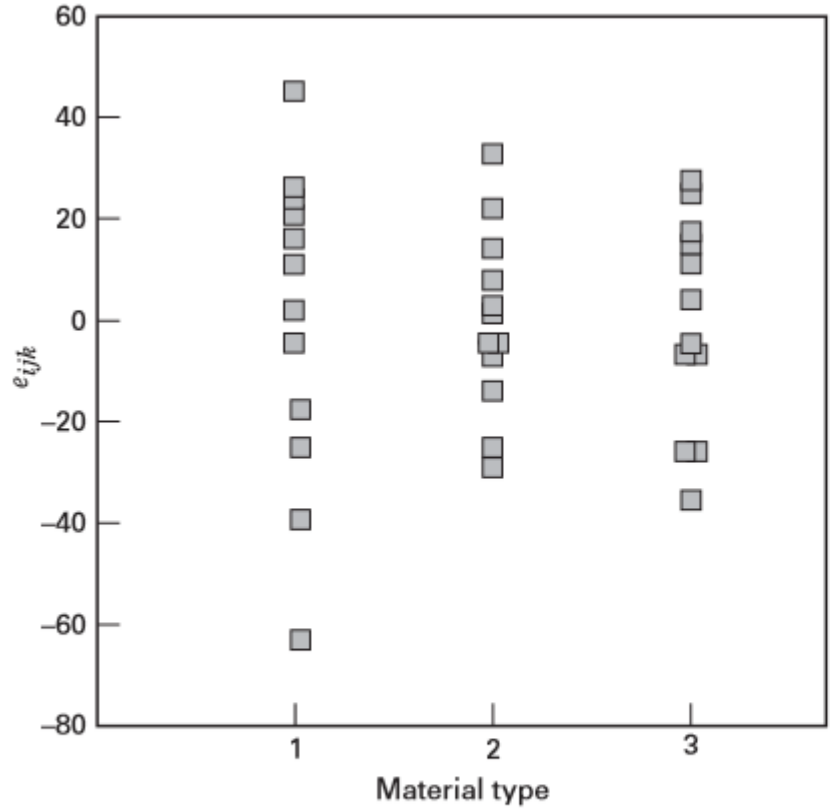
Material Type	Temperature (°F)					
	15		70		125	
1	−4.75	20.25	−23.25	−17.25	−37.50	12.50
	−60.75	45.25	22.75	17.75	24.50	0.50
2	−5.75	32.25	16.25	2.25	−24.50	20.50
	3.25	−29.75	−13.75	−4.75	8.50	−4.50
3	−6.00	−34.00	28.25	−25.75	10.50	18.50
	24.00	16.00	4.25	−6.75	−3.50	−25.50



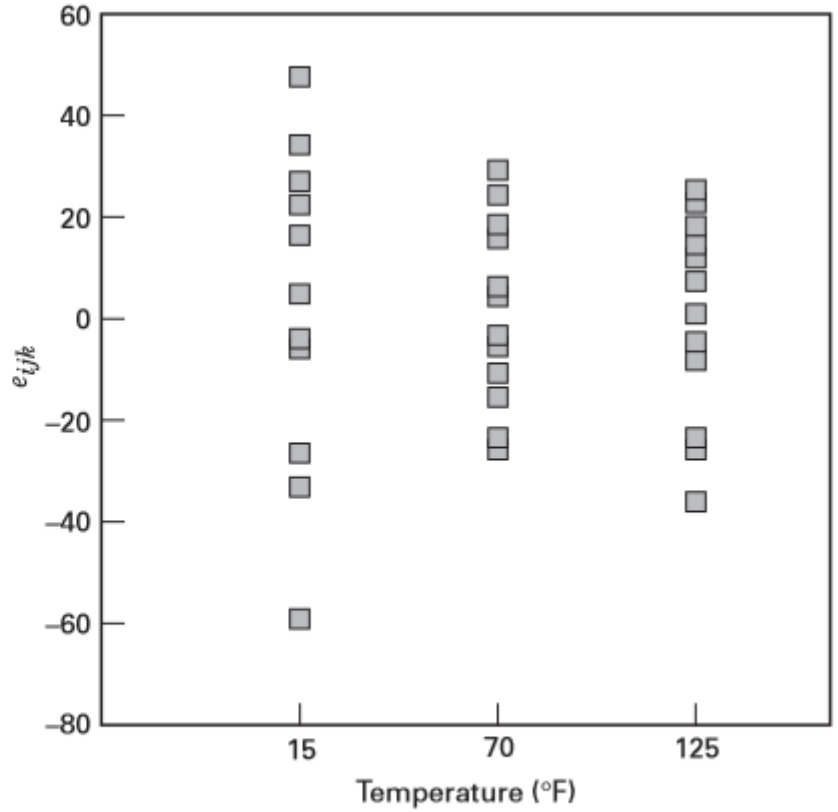
■ **FIGURE 5.11**
Normal probability plot of residuals for Example 5.1



■ **FIGURE 5.12**
Plot of residuals versus $\hat{y}_{ijk}$ for Example 5.1



■ FIGURE 5.13
Plot of residuals versus material type for Example 5.1



■ FIGURE 5.14
Plot of residuals versus temperature for Example 5.1

- From Table 5.6 we see that the 15°F-material type 1 cell contains both extreme residuals (-60.75 and 45.25). These two residuals are primarily responsible for the inequality of variance detected in Figures 5.12, 5.13, and 5.14.

Estimated model parameters

The parameters in the effects model for two-factor factorial

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \qquad (5.13)$$

- Are calculated as

$$\begin{aligned}
\hat{\mu} &= \bar{y}_{...} \\
\hat{\tau}_i &= \bar{y}_{i..} - \bar{y}_{...} \qquad i = 1, 2, \ldots, a \\
\hat{\beta}_j &= \bar{y}_{.j.} - \bar{y}_{...} \qquad j = 1, 2, \ldots, b \\
\widehat{(\tau\beta)}_{ij} &= \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases} \qquad (5.16)
\end{aligned}$$

Choice of sample size

- The operating characteristic curves in Appendix Chart V (shown in the example) in the can be used to assist the experimenter in determining an appropriate sample size (number of replicates, n) for a two-factor factorial design.
- The appropriate value of the parameter Φ^2 and the numerator and denominator degrees of freedom are shown in Table 5.7.
- A very effective way to use these curves is to find the smallest value of Φ^2 corresponding to a specified difference between any two treatment means. For example, if the difference in any two row means is D, then the minimum value of Φ^2 is $\Phi^2 = \frac{nbD^2}{2a\sigma^2}$
- if the difference in any two column means is D, then the minimum value of Φ^2 is $\Phi^2 = \frac{naD^2}{2b\sigma^2}$

■ **TABLE 5.7**

Operating Characteristic Curve Parameters for Chart V of the Appendix for the Two-Factor Factorial, Fixed Effects Model

Factor	Φ^2	Numerator Degrees of Freedom	Denominator Degrees of Freedom
A	$\dfrac{bn\sum_{i=1}^{a}\tau_i^2}{a\sigma^2}$	$a-1$	$ab(n-1)$
B	$\dfrac{an\sum_{j=1}^{b}\beta_j^2}{b\sigma^2}$	$b-1$	$ab(n-1)$
AB	$\dfrac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{\sigma^2[(a-1)(b-1)+1]}$	$(a-1)(b-1)$	$ab(n-1)$

An example

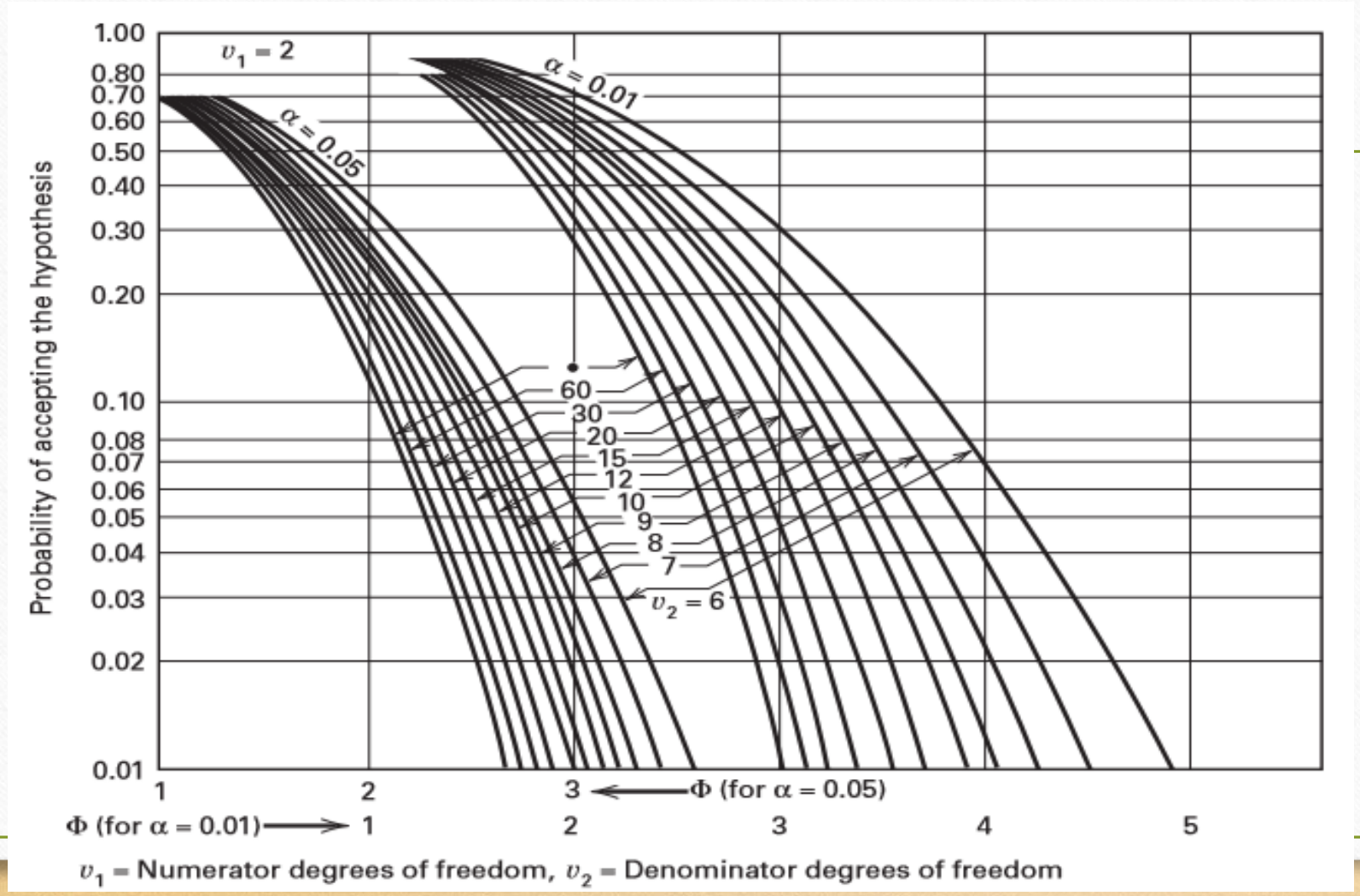
- Consider the battery life data in Example 5.1.
- Suppose that before running the experiment we decide that the null hypothesis should be rejected with a high probability if the difference in mean battery life between any two temperatures is as great as 40 hours.
- Thus a difference of $D = 40$ has engineering significance, and if we assume that the standard deviation of battery life is approximately 25,then Equation 5.18 (sample size based on column mean, mentioned earlier)gives:

$$\Phi^2 = \frac{naD^2}{2b\sigma^2}$$
$$= \frac{n(3)(40)^2}{2(3)(25)^2}$$
$$= 1.28n$$

as the minimum value of Φ^2. Assuming that $\alpha = 0.05$, we can now use Appendix Table V to construct the following display:

n	Φ^2	Φ	ν_1 = Numerator Degrees of Freedom	ν_2 = Error Degrees of Freedom	β
2	2.56	1.60	2	9	0.45
3	3.84	1.96	2	18	0.18
4	5.12	2.26	2	27	0.06

Appendix 5 (corresponding to v_1 and v_2 given)



The Assumption of No Interaction in a Two-Factor Model

Occasionally, an experimenter feels that a **two-factor model without interaction** is appropriate, say

$$y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \qquad \textbf{(5.20)}$$

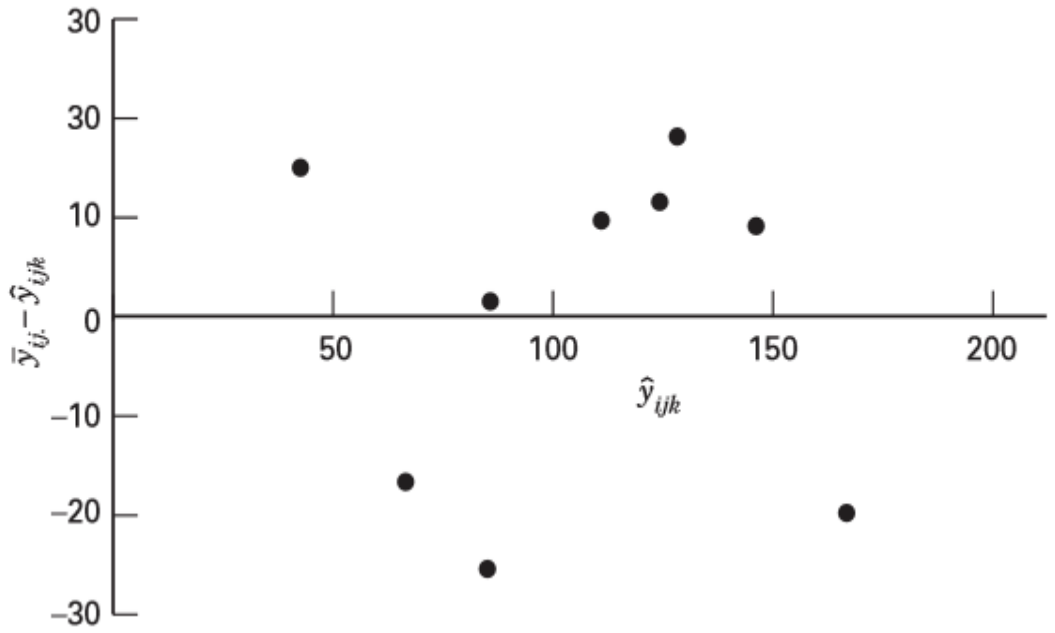
■ **TABLE 5.8**

Analysis of Variance for Battery Life Data Assuming No Interaction

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Material types	10,683.72	2	5,341.86	5.95
Temperature	39,118.72	2	19,559.36	21.78
Error	27,844.52	31	898.21	
Total	77,646.96	35		

- In table 5.18, we assume that the no interaction model applies.

as soon as a residual analysis is performed for these data, it becomes clear that the no-interaction model is inadequate. For the two-factor model without interaction, the fitted values are $\hat{y}_{ijk} = \bar{y}_{i..} + \bar{y}_{.j.} - \bar{y}_{...}$. A plot of $\bar{y}_{ij.} - \hat{y}_{ijk}$ (the cell averages minus the fitted value for that cell) versus the fitted value $\hat{y}_{ijk}$ is shown in Figure 5.15. Now the quantities $\bar{y}_{ij.} - \hat{y}_{ijk}$ may be viewed as the differences between the observed cell means and the estimated cell means assuming no interaction. Any pattern in these quantities is suggestive of the presence of interaction. Figure 5.15 shows a distinct pattern as the quantities $\bar{y}_{ij.} - \hat{y}_{ijk}$ move from positive to negative to positive to negative again. This structure is the result of interaction between material types and temperature.



■ **FIGURE 5.15**
Plot of $\bar{y}_{ij.} - \hat{y}_{ijk}$ versus $\hat{y}_{ijk}$, battery life data

One Observation per Cell

- Occasionally, one encounters a two-factor experiment with only a single replicate, that is, only one observation per cell. If there are two factors and only one observation per cell, the effects model is:

$$y_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ij} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

- Here there are no tests on main effects unless the interaction effect is zero.
- Then the above model becomes:

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases} \qquad (5.22)$$

- A test developed by Tukey (1949a) is helpful in determining whether interaction is present. The procedure assumes that the interaction term is of a particularly simple form, namely, $(\tau\beta)_{ij} = \gamma\tau_i\beta_j$ where γ is an unknown constant.
- We may use a regression approach to test the significance of the interaction term

Computationally, we have

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_ASS_B} \tag{5.23}$$

with one degree of freedom, and

$$SS_{\text{Error}} = SS_{\text{Residual}} - SS_N \tag{5.24}$$

with $(a-1)(b-1)-1$ degrees of freedom. To test for the presence of interaction, we compute

$$F_0 = \frac{SS_N}{SS_{\text{Error}}/[(a-1)(b-1)-1]} \tag{5.25}$$

If $F_0 > F_{\alpha,1,(a-1)(b-1)-1}$, the hypothesis of no interaction must be rejected.

Example

- The impurity present in a chemical product is affected by two factors—pressure and temperature. The data from a single replicate of a factorial experiment are shown in Table 5.10

■ TABLE 5.10
Impurity Data for Example 5.2

Temperature	Pressure					
(°F)	25	30	35	40	45	$y_{i.}$
100	5	4	6	3	5	23
125	3	1	4	2	3	13
150	1	1	3	1	2	8
$y_{.j}$	9	6	13	6	10	$44 = y_{..}$

The sums of squares are

$$SS_A = \frac{1}{b}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{ab}$$

$$= \frac{1}{5}[23^2 + 13^2 + 8^2] - \frac{44^2}{(3)(5)} = 23.33$$

$$SS_B = \frac{1}{a}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{ab}$$

$$= \frac{1}{3}[9^2 + 6^2 + 13^2 + 6^2 + 10^2] - \frac{44^2}{(3)(5)} = 11.60$$

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab}$$

$$= 166 - 129.07 = 36.93$$

and

$$SS_{\text{Residual}} = SS_T - SS_A - SS_B$$

$$= 36.93 - 23.33 - 11.60 = 2.00$$

The sum of squares for nonadditivity is computed from Equation 5.23 as follows:

$$\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} = (5)(23)(9) + (4)(23)(6) + \cdots + (2)(8)(10) = 7236$$

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_ASS_B}$$

$$= \frac{[7236 - (44)(23.33 + 11.60 + 129.07)]^2}{(3)(5)(23.33)(11.60)}$$

$$= \frac{[20.00]^2}{4059.42} = 0.0985$$

and the error sum of squares is, from Equation 5.24,

$$SS_{\text{Error}} = SS_{\text{Residual}} - SS_N$$

$$= 2.00 - 0.0985 = 1.9015$$

The complete ANOVA is summarized in Table 5.11. The test statistic for nonadditivity is $F_0 = 0.0985/0.2716 = 0.36$, so we conclude that there is no evidence of interaction in these data. The main effects of temperature and pressure are significant.

■ TABLE 5.11

Analysis of Variance for Example 5.2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Temperature	23.33	2	11.67	42.97	0.0001
Pressure	11.60	4	2.90	10.68	0.0042
Nonadditivity	0.0985	1	0.0985	0.36	0.5674
Error	1.9015	7	0.2716		
Total	36.93	14			

The General Factorial Design

- The results for the two-factor factorial design may be extended to the general case where there are a levels of factor A, b levels of factor B, c levels of factor C, and so on, arranged in a factorial experiment.
- In general, there will be abc . . . n total observations if there are n replicates of the complete experiment.
- we must have at least two replicates ($n \geq 2$) to determine a sum of squares due to error if all possible interactions are included in the model.

- If all factors in the experiment are fixed, we may easily formulate and test hypotheses about the main effects and interactions using the ANOVA.
- For a fixed effects model, test statistics for each main effect and interaction may be constructed by dividing the corresponding mean square for the effect or interaction by the mean square error.
- All of these F tests will be upper-tail, one-tail tests.
- The number of degrees of freedom for any main effect is the number of levels of the factor minus one, and the number of degrees of freedom for an interaction is the product of the number of degrees of freedom associated with the individual components of the interaction.

consider the **three-factor analysis of variance model**:

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk}$$

$$+ (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \\ l = 1, 2, \ldots, n \end{cases}$$

- Assuming that A, B, and C are fixed, we get the following table for ANOVA:

The Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square	F_0
A	SS_A	$a-1$	MS_A	$\sigma^2 + \frac{bcn\sum \tau_i^2}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn\sum \beta_j^2}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn\sum \gamma_k^2}{c-1}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn\sum\sum (\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an\sum\sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n\sum\sum\sum (\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

Manual computation formulas

The total sum of squares is found in the usual way as

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{n} y_{ijkl}^2 - \frac{y_{....}^2}{abcn}$$

The sums of squares for the main effects are found from the totals for factors $A(y_{i...})$, $B(y_{.j..})$, and $C(y_{..k.})$ as follows:

$$SS_A = \frac{1}{bcn}\sum_{i=1}^{a} y_{i...}^2 - \frac{y_{....}^2}{abcn} \qquad \textbf{(5.28)}$$

$$SS_B = \frac{1}{acn}\sum_{j=1}^{b} y_{.j..}^2 - \frac{y_{....}^2}{abcn} \qquad \textbf{(5.29)}$$

$$SS_C = \frac{1}{abn}\sum_{k=1}^{c} y_{..k.}^2 - \frac{y_{....}^2}{abcn} \qquad \textbf{(5.30)}$$

To compute the two-factor interaction sums of squares, the totals for the $A \times B$, $A \times C$, and $B \times C$ cells are needed. It is frequently helpful to collapse the original data table into three two-way tables to compute these quantities. The sums of squares are found from

$$SS_{AB} = \frac{1}{cn}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B$$

$$= SS_{\text{Subtotals}(AB)} - SS_A - SS_B \tag{5.31}$$

$$SS_{AC} = \frac{1}{bn}\sum_{i=1}^{a}\sum_{k=1}^{c} y_{i.k.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_C$$

$$= SS_{\text{Subtotals}(AC)} - SS_A - SS_C \tag{5.32}$$

and

$$SS_{BC} = \frac{1}{an}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{.jk.}^2 - \frac{y_{....}^2}{abcn} - SS_B - SS_C$$

$$= SS_{\text{Subtotals}(BC)} - SS_B - SS_C \tag{5.33}$$

Note that the sums of squares for the two-factor subtotals are found from the totals in each two-way table. The three-factor interaction sum of squares is computed from the three-way cell totals $\{y_{ijk.}\}$ as

$$SS_{ABC} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC} \tag{5.34a}$$

$$= SS_{\text{Subtotals}(ABC)} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC} \tag{5.34b}$$

The error sum of squares may be found by subtracting the sum of squares for each main effect and interaction from the total sum of squares or by

$$SS_E = SS_T - SS_{\text{Subtotals}(ABC)} \tag{5.35}$$

An example

A soft drink bottler is interested in obtaining more uniform fill heights in the bottles produced by his manufacturing process. The filling machine theoretically fills each bottle to the correct target height, but in practice, there is variation around this target, and the bottler would like to understand the sources of this variability better and eventually reduce it.

The process engineer can control three variables during the filling process: the percent carbonation (A), the operating pressure in the filler (B), and the bottles produced per minute or the line speed (C). The pressure and speed are easy to control, but the percent carbonation is more difficult to control during actual manufacturing because it varies with product temperature. However, for purposes of an experiment, the engineer can control carbonation at three levels: 10, 12, and 14 percent. She chooses two levels for pressure (25 and 30 psi) and two levels for line speed (200 and 250 bpm). She decides to run two replicates of a factorial design in these three factors, with all 24 runs taken in random order. The response variable observed is the average deviation from the target fill height observed in a production run of bottles at each set of conditions. The data that resulted from this experiment are shown in Table 5.13. Positive deviations are fill heights above the target, whereas negative deviations are fill heights below the target. The circled numbers in Table 5.13 are the three-way cell totals $y_{ijk\cdot}$.

Fill Height Deviation Data for Example 5.3

Percent Carbonation (A)	Operating Pressure (B): 25 psi, Line Speed (C): 200		Operating Pressure (B): 25 psi, Line Speed (C): 250		Operating Pressure (B): 30 psi, Line Speed (C): 200		Operating Pressure (B): 30 psi, Line Speed (C): 250		$y_{i\cdots}$
10	−3 −1	(−4)	−1 0	(−1)	−1 0	(−1)	1 1	(2)	−4
12	0 1	(1)	2 1	(3)	2 3	(5)	6 5	(11)	20
14	5 4	(9)	7 6	(13)	7 9	(16)	10 11	(21)	59
$B \times C$ Totals $y_{jk.}$	6		15		20		34		$75 = y_{\cdots}$
$y_{j\cdots}$	21				54				

$A \times B$ Totals $y_{ij\cdots}$

A	B: 25	B: 30
10	−5	1
12	4	16
14	22	37

$A \times C$ Totals $y_{i\cdot k\cdot}$

A	C: 200	C: 250
10	−5	1
12	6	14
14	25	34

- Table 5.13

The total corrected sum of squares is found from Equation 5.27 as

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{n} y_{ijkl}^2 - \frac{y_{....}^2}{abcn}$$

$$= 571 - \frac{(75)^2}{24} = 336.625$$

and the sums of squares for the main effects are calculated from Equations 5.28, 5.29, and 5.30 as

$$SS_{\text{Carbonation}} = \frac{1}{bcn}\sum_{i=1}^{a} y_{i...}^2 - \frac{y_{....}^2}{abcn}$$

$$= \frac{1}{8}[(-4)^2 + (20)^2 + (59)^2] - \frac{(75)^2}{24} = 252.750$$

$$SS_{\text{Pressure}} = \frac{1}{acn}\sum_{j=1}^{b} y_{.j..}^2 - \frac{y_{....}^2}{abcn}$$

$$= \frac{1}{12}[(21)^2 + (54)^2] - \frac{(75)^2}{24} = 45.375$$

and

$$SS_{\text{Speed}} = \frac{1}{abn}\sum_{k=1}^{c} y_{..k.}^2 - \frac{y_{....}^2}{abcn}$$

$$= \frac{1}{12}[(26)^2 + (49)^2] - \frac{(75)^2}{24} = 22.042$$

To calculate the sums of squares for the two-factor interactions, we must find the two-way cell totals. For example, to find the carbonation–pressure or AB interaction, we need the totals for the $A \times B$ cells $\{y_{ij..}\}$ shown in Table 5.13. Using Equation 5.31, we find the sums of squares as

$$SS_{AB} = \frac{1}{cn}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B$$

$$= \frac{1}{4}[(-5)^2 + (1)^2 + (4)^2 + (16)^2 + (22)^2 + (37)^2] - \frac{(75)^2}{24} - 252.750 - 45.375$$

$$= 5.250$$

The carbonation–speed or AC interaction uses the $A \times C$ cell totals $\{y_{i.k.}\}$ shown in Table 5.13 and Equation 5.32:

$$SS_{AC} = \frac{1}{bn}\sum_{i=1}^{a}\sum_{k=1}^{c} y_{i.k.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_C$$

$$= \frac{1}{4}[(-5)^2 + (1)^2 + (6)^2 + (14)^2 + (25)^2 + (34)^2] - \frac{(75)^2}{24} - 252.750 - 22.042$$

$$= 0.583$$

The pressure–speed or BC interaction is found from the $B \times C$ cell totals $\{y_{.jk.}\}$ shown in Table 5.13 and Equation 5.33:

$$SS_{BC} = \frac{1}{an}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{.jk.}^2 - \frac{y_{....}^2}{abcn} - SS_B - SS_C$$

$$= \frac{1}{6}[(6)^2 + (15)^2 + (20)^2 + (34)^2] - \frac{(75)^2}{24} - 45.375 - 22.042$$

$$= 1.042$$

The three-factor interaction sum of squares is found from the $A \times B \times C$ cell totals $\{y_{ijk.}\}$, which are circled in Table 5.13. From Equation 5.34a, we find

$$\begin{aligned} SS_{ABC} &= \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2 - \frac{y_{....}^2}{abcn} - SS_A - SS_B - SS_C \\ &\quad - SS_{AB} - SS_{AC} - SS_{BC} \\ &= \frac{1}{2}\left[(-4)^2 + (-1)^2 + (-1)^2 + \cdots + (16)^2 + (21)^2\right] \\ &\quad - \frac{(75)^2}{24} - 252.750 - 45.375 - 22.042 \\ &\quad - 5.250 - 0.583 - 1.042 \\ &= 1.083 \end{aligned}$$

Finally, noting that

$$SS_{\text{Subtotals}(ABC)} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2 - \frac{y_{....}^2}{abcn} = 328.125$$

we have

$$\begin{aligned} SS_E &= SS_T - SS_{\text{Subtotals}(ABC)} \\ &= 336.625 - 328.125 \\ &= 8.500 \end{aligned}$$

The ANOVA is summarized in Table 5.14. We see that the percentage of carbonation, operating pressure, and line speed significantly affect the fill volume. The carbonation-pressure interaction F ratio has a P-value of 0.0558, indicating some interaction between these factors.

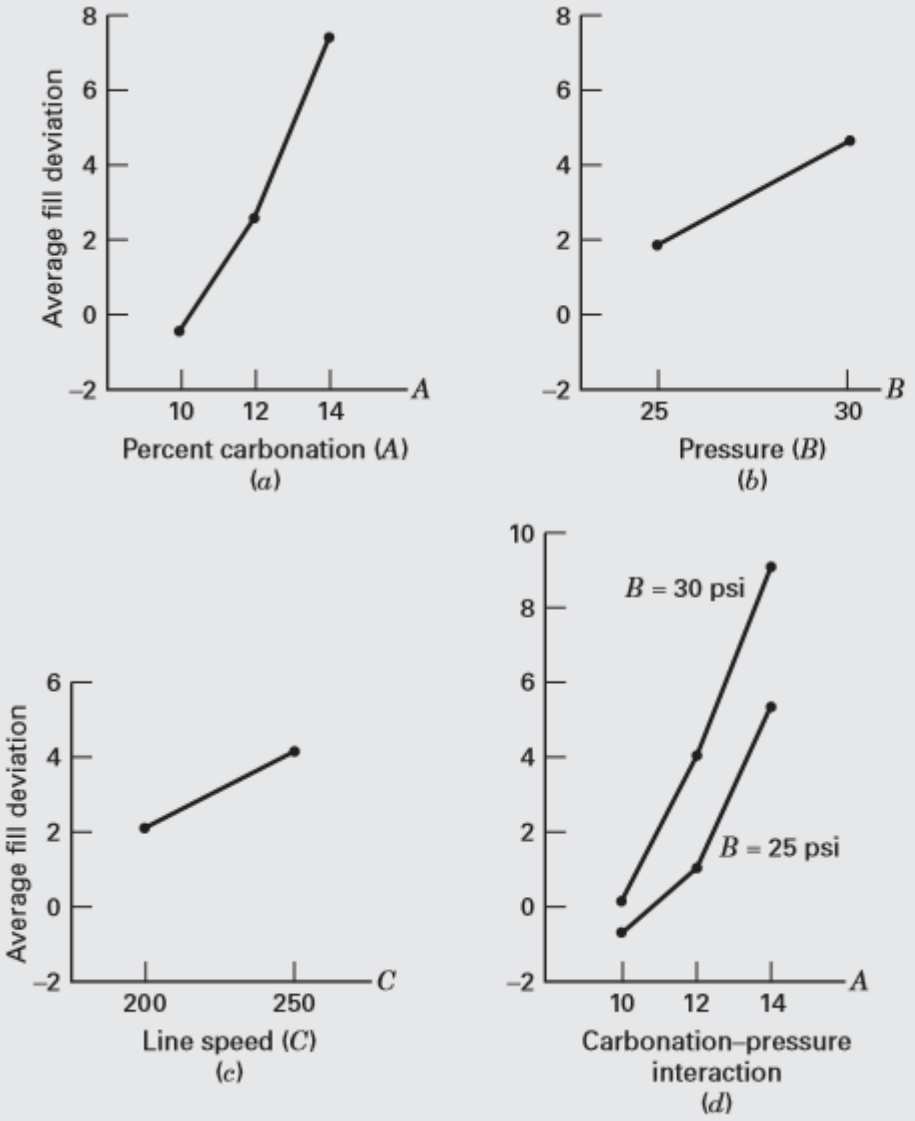
The next step should be an analysis of the residuals from this experiment. We leave this as an exercise for the reader but point out that a normal probability plot of the residuals and the other usual diagnostics do not indicate any major concerns.

■ **TABLE 5.14**
Analysis of Variance for Example 5.3

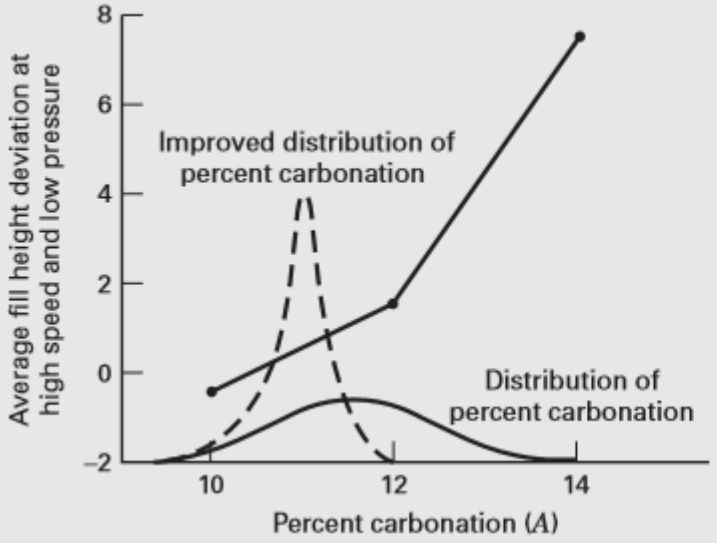
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Percentage of carbonation (A)	252.750	2	126.375	178.412	<0.0001
Operating pressure (B)	45.375	1	45.375	64.059	<0.0001
Line speed (C)	22.042	1	22.042	31.118	0.0001
AB	5.250	2	2.625	3.706	0.0558
AC	0.583	2	0.292	0.412	0.6713
BC	1.042	1	1.042	1.471	0.2485
ABC	1.083	2	0.542	0.765	0.4867
Error	8.500	12	0.708		
Total	336.625	23			

To assist in the practical interpretation of this experiment, Figure 5.16 presents plots of the three main effects and the *AB* (carbonation–pressure) interaction. The main effect plots are just graphs of the marginal response averages at the levels of the three factors. Notice that all three variables have *positive* main effects; that is, increasing the variable moves the average deviation from the fill target upward. The interaction between carbonation and pressure is fairly small, as shown by the similar shape of the two curves in Figure 5.16*d*.

Because the company wants the average deviation from the fill target to be close to zero, the engineer decides to recommend the low level of operating pressure (25 psi) and the high level of line speed (250 bpm, which will maximize the production rate). Figure 5.17 plots the average observed deviation from the target fill height at the three different carbonation levels for this set of operating conditions. Now the carbonation level cannot presently be perfectly controlled in the manufacturing process, and the normal distribution shown with the solid curve in Figure 5.17 approximates the variability in the carbonation levels presently experienced. As the process is impacted by the values of the carbonation level drawn from this distribution, the fill heights will fluctuate considerably. This variability in the fill heights could be reduced if the distribution of the carbonation level values followed the normal distribution shown with the dashed line in Figure 5.17. Reducing the standard deviation of the carbonation level distribution was ultimately achieved by improving temperature control during manufacturing.



■ **FIGURE 5.16**
Main effects and interaction plots for Example 5.3. (*a*) Percentage of carbonation (*A*). (*b*) Pressure (*B*). (*c*) Line speed (*C*). (*d*) Carbonation–pressure interaction



■ **FIGURE 5.17**
Average fill height deviation at high speed and low pressure for different carbonation levels

Fitting Response Curves and Surfaces

- The ANOVA always treats all of the factors in the experiment as if they were qualitative or categorical.
- However, many experiments involve at least one quantitative factor.
- It can be useful to fit a response curve to the levels of a quantitative factor so that the experimenter has an equation that relates the response to the factor.
- This equation might be used for interpolation, that is, for predicting the response at factor levels between those actually used in the experiment.
- When at least two factors are quantitative, we can fit a response surface for predicting yat various combinations of the design factors.
- In general, linear regression methods are used to fit these models to the experimental data.

An example

Consider the battery life experiment described in Example 5.1. The factor temperature is quantitative, and the material type is qualitative. Furthermore, there are three levels of temperature. Consequently, we can compute a linear and a quadratic temperature effect to study how temperature affects the battery life. Table 5.15 presents condensed output from Design-Expert for this experiment and assumes that temperature is quantitative and material type is qualitative.

The ANOVA in Table 5.15 shows that the "model" source of variability has been subdivided into several components. The components "A" and "A^2" represent the linear and quadratic effects of temperature, and "B" represents the main effect of the material type factor. Recall that material type is a qualitative factor with three levels. The terms "AB" and "A^2B" are the interactions of the linear and quadratic temperature factor with material type.

The P-values indicate that A^2 and AB are not significant, whereas the A^2B term is significant. Often we think about removing nonsignificant terms or factors from a model, but in this case, removing A^2 and AB and retaining A^2B will result in a model that is not **hierarchical**. The **hierarchy principle** indicates that if a model contains a high-order term (such as A^2B), it should also contain all of the lower order terms that compose it (in this case A^2 and AB). Hierarchy promotes a type of internal consistency in a model, and many statistical model builders rigorously follow the principle. However, hierarchy is not always a good idea, and many models actually work better as prediction equations without including the nonsignificant terms that promote hierarchy. For more information, see the supplemental text material for this chapter.

The computer output also gives model coefficient estimates and a final prediction equation for battery life in coded factors. In this equation, the levels of temperature are $A = -1, 0, +1$, respectively, when temperature is at the low, middle, and high levels (15, 70, and 125°C). The variables $B[1]$ and $B[2]$ are coded **indicator variables** that are defined as follows:

	Material Type		
	1	**2**	**3**
$B[1]$	1	0	−1
$B[2]$	0	1	−1

There are also prediction equations for battery life in terms of the actual factor levels. Notice that because material type is a qualitative factor there is an equation for predicted life as a function of temperature for each material type. Figure 5.18 shows the response curves generated by these three prediction equations. Compare them to the two-factor interaction graph for this experiment in Figure 5.9.

■ TABLE 5.15

Design-Expert Output for Example 5.4

Response: Life In Hours

ANOVA for Response Surface Reduced Cubic Model

Analysis of Variance Table [Partial Sum of Squares]

Source	Sum of Squares	DF	Mean Square	*F* Value	Prov > *F*	
Model	59416.22	8	7427.03	11.00	<0.0001	significant
A	*39042.67*	*1*	*39042.67*	*57.82*	<0.0001	
B	*10683.72*	*2*	*5341.86*	*7.91*	*0.0020*	
A^2	*76.06*	*1*	*76.06*	*0.11*	*0.7398*	
AB	*2315.08*	*2*	*1157.54*	*1.71*	*0.1991*	
A^2B	*7298.69*	*2*	*3649.35*	*5.40*	*0.0106*	
Residual	18230.75	27	675.21			
Lack of Fit	*0.000*	*0*				
Pure Error	*18230.75*	*27*	*675.21*			
Cor Total	77646.97	35				

Std. Dev.	25.98	R-Squared	0.7652
Mean	105.53	Adj R-Squared	0.6956

C.V.	24.62		Pred R-Squared	0.5826	
PRESS	32410.22		Adeq Precision	8.178	

Term	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	107.58	1	7.50	92.19	122.97	
A-Temp	−40.33	1	5.30	−51.22	−29.45	1.00
B[1]	−50.33	1	10.61	−72.10	−28.57	
B[2]	12.17	1	10.61	−9.60	33.93	
A^2	−3.08	1	9.19	−21.93	15.77	1.00
AB[1]	1.71	1	7.50	−13.68	17.10	
AB[2]	−12.79	1	7.50	−28.18	2.60	
A^2B[1]	41.96	1	12.99	15.30	68.62	
A^2B[2]	−14.04	1	12.99	−40.70	12.62	

Final Equation in Terms of Coded Factors:

Life =
+107.58
−40.33 *A
−50.33 *B[1]
+12.17 *B[2]
−3.08 *A^2
+1.71 *AB[1]
−12.79 *AB[2]
+41.96 *A^2B[1]
−14.04 *A^2[2]

Final Equation in Terms of Actual Factors:

Material Type 1

Life =
+169.38017
−2.48860 *Temp
+0.012851 *$Temp^2$

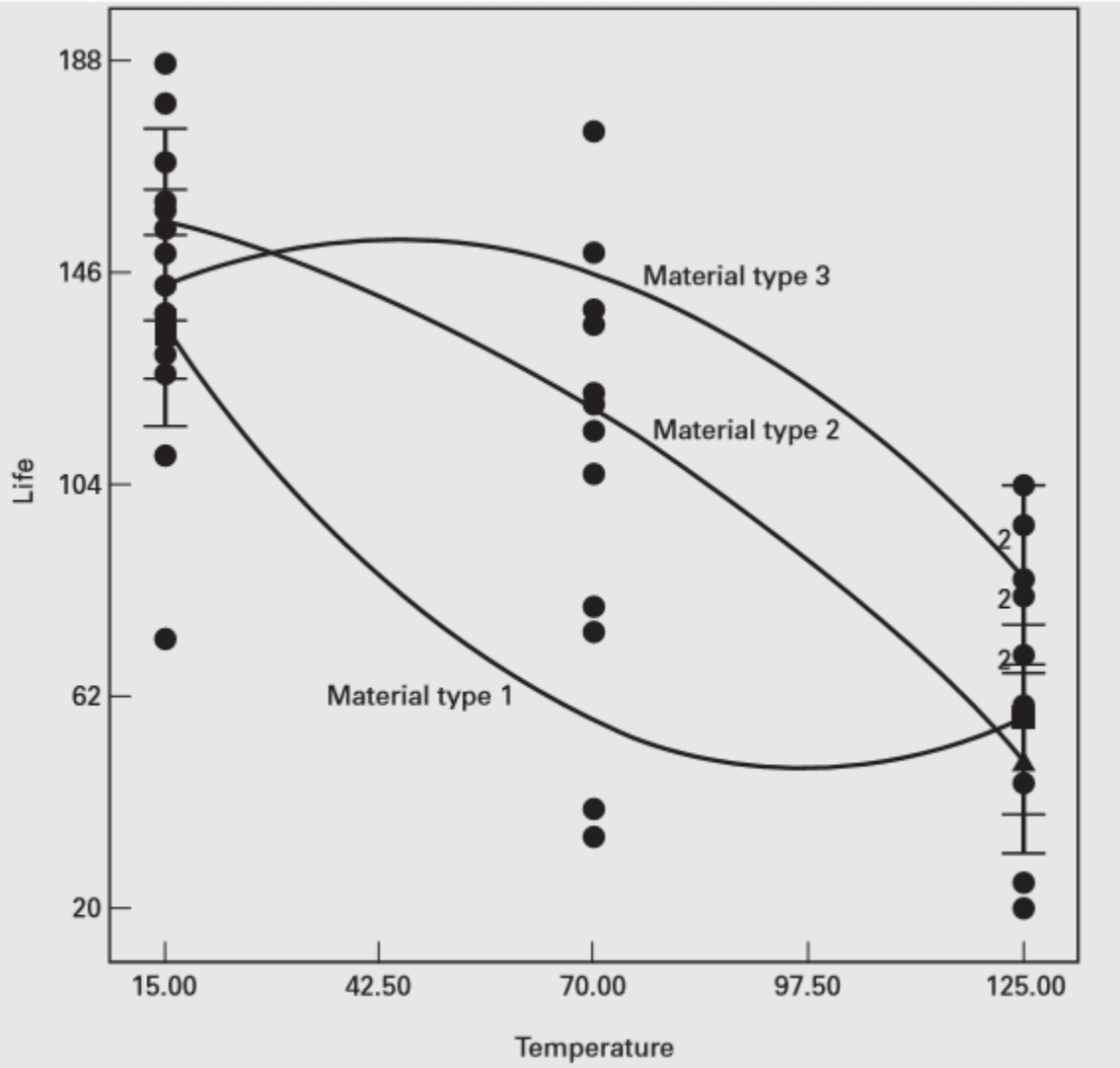
Material Type 2

Life =
+159.62397
−0.17901 *Temp
+0.41627 *$Temp^2$

Material Type 3

Life =
+132.76240
+0.89264 *Temp
−0.43218 *$Temp^2$

■ FIGURE 5.18
Predicted life as a function of temperature for the three material types, Example 5.4



Blocking in a Factorial Design

- We have discussed factorial designs in the context of a completely randomized experiment. Sometimes, it is not feasible or practical to completely randomize all of the runs in a factorial.
- For example, the presence of a nuisance factor may require that the experiment be run in blocks.

Consider a factorial experiment with two factors (A and B) and n replicates. The linear statistical model for this design is

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \qquad (5.36)$$

τ_i, β_j, and $(\tau\beta)_{ij}$ represent the effects of factors A, B, and the AB interaction.

- suppose that to run this experiment a particular raw material is required. This raw material is available in batches that are not large enough to allow all abn treatment combinations to be run from the same batch.
- However, if a batch contains enough material for ab observations, then an alternative design is to run each of the n replicates using a separate batch of raw material.
- Consequently, the batches of raw material represent a randomization restriction or a block, and a single replicate of a complete factorial experiment is run within each block.

The effects model for this new design is

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \delta_k + \epsilon_{ijk} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \qquad (5.37)$$

where δ_k is the effect of the kth block. Of course, within a block the order in which the treatment combinations are run is completely randomized.

- The model (Equation 5.37) assumes that interaction between blocks and treatments is negligible.
- The analysis of variance is outlined in the following table:

Analysis of Variance for a Two-Factor Factorial in a Randomized Complete Block

Source of Variation	Sum of Squares	Degrees of Freedom	Expected Mean Square	F_0
Blocks	$\frac{1}{ab}\sum_k y_{..k}^2 - \frac{y_{...}^2}{abn}$	$n-1$	$\sigma^2 + ab\sigma_\delta^2$	
A	$\frac{1}{bn}\sum_i y_{i..}^2 - \frac{y_{...}^2}{abn}$	$a-1$	$\sigma^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	$\frac{1}{an}\sum_j y_{.j.}^2 - \frac{y_{...}^2}{abn}$	$b-1$	$\sigma^2 + \frac{an\sum \beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
AB	$\frac{1}{n}\sum_i\sum_j y_{ij.}^2 - \frac{y_{...}^2}{abn} - SS_A - SS_B$	$(a-1)(b-1)$	$\sigma^2 + \frac{n\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
Error	Subtraction	$(ab-1)(n-1)$	σ^2	
Total	$\sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{abn}$	$abn-1$		

The ANOVA estimator of the variance component for blocks σ_δ^2, is

$$\sigma_\delta^2 = \frac{MS_{Blocks} - MS_E}{ab}$$

- The randomization was restricted to within a batch of raw material. In practice, a variety of phenomena may cause randomization restrictions, such as time and operators.
- For example, if we could not run the entire factorial experiment on one day, then the experimenter could run a complete replicate on day 1, a second replicate on day 2, and so on. Consequently, each day would be a block.

An example

An engineer is studying methods for improving the ability to detect targets on a radar scope. Two factors she considers to be important are the amount of background noise, or "ground clutter," on the scope and the type of filter placed over the screen. An experiment is designed using three levels of ground clutter and two filter types. We will consider these as fixed-type factors. The experiment is performed by randomly selecting a treatment combination (ground clutter level and filter type) and then introducing a signal representing the target into the scope. The intensity of this target is increased until the operator observes it. The intensity level at detection is then measured as the response variable. Because of operator availability, it is convenient to select an operator and keep him or her at the scope until all the necessary runs have been made. Furthermore, operators differ in their skill and ability to use the scope. Consequently, it seems logical to use the operators as blocks. Four operators are randomly selected. Once an operator is chosen, the order in which the six treatment combinations are run is randomly determined. Thus, we have a 3×2 factorial experiment run in a randomized complete block. The data are shown in Table 5.21.

■ **TABLE 5.21**
Intensity Level at Target Detection

Operators (blocks)	1		2		3		4	
Filter Type	**1**	**2**	**1**	**2**	**1**	**2**	**1**	**2**
Ground clutter								
Low	90	86	96	84	100	92	92	81
Medium	102	87	106	90	105	97	96	80
High	114	93	112	91	108	95	98	83

The linear model for this experiment is

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \delta_k + \epsilon_{ijk} \qquad \begin{cases} i = 1, 2, 3 \\ j = 1, 2 \\ k = 1, 2, 3, 4 \end{cases}$$

where τ_i represents the ground clutter effect, β_j represents the filter type effect, $(\tau\beta)_{ij}$ is the interaction, δ_k is the block effect, and ϵ_{ijk} is the NID(0, σ^2) error component. The sums of squares for ground clutter, filter type, and their interaction are computed in the usual manner. The sum of squares due to blocks is found from the operator totals $\{y_{..k}\}$ as follows:

$$\begin{aligned} SS_{\text{Blocks}} &= \frac{1}{ab}\sum_{k=1}^{n} y_{..k}^2 - \frac{y_{...}^2}{abn} \\ &= \frac{1}{(3)(2)}[(572)^2 + (579)^2 + (597)^2 + (530)^2] \\ &\quad - \frac{(2278)^2}{(3)(2)(4)} \\ &= 402.17 \end{aligned}$$

The complete ANOVA for this experiment is summarized in Table 5.22. The presentation in Table 5.22 indicates that all effects are tested by dividing their mean squares by the mean square error. Both ground clutter level and filter type are significant at the 1 percent level, whereas their interaction is significant only at the 10 percent level. Thus, we conclude that both ground clutter level and the type of scope filter used affect the operator's ability to detect the target, and there is

■ **TABLE 5.22**
Analysis of Variance for Example 5.6

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Ground clutter (G)	335.58	2	167.79	15.13	0.0003
Filter type (F)	1066.67	1	1066.67	96.19	<0.0001
GF	77.08	2	38.54	3.48	0.0573
Blocks	402.17	3	134.06		
Error	166.33	15	11.09		
Total	2047.83	23			